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Unsteady Duct flows

Unsteady plane shear flows are experienced by fluids where the driving forces are suddenly or continuously changed. A fluid initially at rest will for example respond to a sudden increase in the pressure gradient and the volume flow through the duct will then increase monotonically until steady state is reached.

Assumption

1) 1-D flow

2) fully developed

3) the pipe axial velocity $u = u(r, t)$

4) $U_r = 0$

5) $U_\infty = 0$

continuity equation

$$\rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r U_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho U_\theta) + \frac{\partial}{\partial z} (\rho U_z) = 0$$

$$\rightarrow \frac{1}{r} * \frac{\partial}{\partial r} (r U_r) + \frac{1}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z} = 0$$

$$\frac{1}{r} \left[r \frac{\partial U_r}{\partial r} + U \frac{\partial r}{\partial r} \right] + \frac{1}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z} = 0$$

$$\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z} = 0$$

(u) 0 (u) 0 0

$$\Rightarrow \frac{\partial U_z}{\partial z} = 0$$

$\Rightarrow U \neq f(x)$

$U \neq f(\theta)$ axisymetric flow

$\Rightarrow U_z = f(r)$ only

→ z-momentum equation

$$\rho\left(\frac{\partial U_z}{\partial t} + U_r\frac{\partial U_z}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_z}{\partial \theta} + U_z\frac{\partial U_z}{\partial z}\right) = -\frac{\partial P}{\partial z}$$

(4) 0 (5) 0 (2) 0

(geometry) 0

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U_z}{\partial \theta^2} + \frac{\partial^2 U_z}{\partial z^2}\right] + \rho g_z$$

(1) 0 0 (continuity)

$$\Rightarrow \frac{\partial U_z}{\partial t} + = \frac{-1}{\rho}\frac{\partial P}{\partial z} + \frac{\mu}{\rho}\left[\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r}\frac{\partial U_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U_z}{\partial \theta^2} + \frac{\partial^2 U_z}{\partial z^2}\right] + g_z$$

$$\Rightarrow \frac{\partial U_z}{\partial t} = \frac{-1}{\rho}\frac{\partial P}{\partial z} + \frac{\mu}{\rho}\left[\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r}\frac{\partial U_z}{\partial r}\right]$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{\mu}{\rho}\left[\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r}\frac{\partial U_z}{\partial r}\right] + \frac{\partial U_z}{\partial t}$$

$$\frac{\partial P}{\partial z} = \mu\left[\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r}\frac{\partial U_z}{\partial r}\right] - \rho\frac{\partial U_z}{\partial t}$$

Starting flow in circular pipes

$\rightarrow u = u_{max}\left(1 - r^{*2}\right)$, $r^* = r/r_0$

$\rightarrow$ boundary condition $u(r,t)$

$u(r,0) = 0 \longrightarrow$ initial condition

$u(r_0,t) = 0 \longrightarrow$ no slip condition

$$u(r,t) = U_{ss}(r) + v(r,t)$$

$$= U_{ss}(r) + u'(r,t)$$

but $U_s = u_m (1 - r^{*2})$

$$U_m = \frac{-r_o^2}{4\mu} \frac{dP}{dx}$$

Using B.C. + initial condition

$$\Rightarrow \quad u_{(r,t)} = -\frac{\partial P}{\cancel{\partial z}\,\partial x}\,\frac{R^2}{4\mu}\left(1-\frac{r^2}{R^2}\right) + u'(r,t)$$

$$\frac{du}{\delta t} = u_{max}\,(-2r) + \frac{\partial u'}{\partial r}$$

$$\frac{\delta^2 u}{\partial r^2} = -2\,U_{max} + \frac{\delta^2 u'}{\partial r^2}$$

using two equation

in momentum-equation

$$\frac{\partial u'}{\partial t} = \frac{-1}{\rho}\frac{\partial P}{\partial x} + \nu\left(-2U_{max} + \frac{\partial^2 u'}{\partial r^2} - 2U_{max} + \frac{1}{r}\frac{\partial u'}{\partial r}\right)$$

$$\Rightarrow \frac{\partial u'}{\partial t} = \frac{-1}{\rho}\frac{\partial P}{\partial x} + \nu\left(-4U_{max} + \frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right)$$

the homogenuous part

$$\frac{1}{\nu}\frac{\partial u'}{\partial t} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{\partial u'}{\partial r} = 0 \quad \text{------} \quad (*)$$

let $u'_{(r,t)} = RT$

seperation of variabels

$$\frac{\partial u'}{\partial t} = R T'$$

$$\frac{\partial u'}{\partial R} = R' T$$

$$\frac{\partial^2 u'}{\partial R^2} = R'' T$$

using equation (*)

$$\frac{1}{\nu} R T' - \frac{1}{r} R' T - R'' T = 0 \qquad / RT$$

$$\frac{1}{\nu} \frac{T'}{T} - \frac{1}{r} \frac{R'}{R} - \frac{R''}{R} = 0$$

$$\Rightarrow \frac{1}{\nu} \frac{T'}{T} = + \frac{1}{r} \frac{R'}{R} + \frac{R''}{R} = -\lambda$$

$$\Rightarrow \frac{T'}{T} = -\lambda \nu \qquad \Rightarrow \ln T = -\nu \lambda^2 t$$

$$\Rightarrow T = c_0 e^{-\lambda^2 \nu T}$$

2

The second solution

$$\frac{1}{r}\frac{R'}{R} + \frac{R''}{R} + \lambda^2 = 0$$ ----- this equation is bessel function zero order

$$r^2 R'' + r R' + R \lambda^2 r^2 = 0$$

$$\rightarrow R(r) = C_1 \, J_0(\lambda r) + C_2 \, Y_0(\lambda r)$$

$$\rightarrow u(r,t) = e^{-\nu \lambda t} \left(A \, J_0(\lambda r) + B \, Y_0(\lambda r)\right)$$

$\rightarrow u(r,t) = e^{-\nu\lambda t}\left(A\, J_0(\lambda r) + B\, Y_0(\lambda r)\right)$

$\rightarrow$ Bessel equation in general

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$\rightarrow$ for zero order

$$x^2 y'' + x y' + x^2 y = 0$$

$$\Rightarrow y = C_1 J_0 + C_2 Y_0$$

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r}}{(r!)^2}$$

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+1}}{(r+1)\,(r!)^2}$$

B c's

$J_0 \to 1 \qquad / \; Y_0 \to \infty$

$\to \; u(0,t) = e^{-\nu \lambda^2 r} \left(A\, J_0(0) + B\, Y_0(0) \right) = 0$

$\Rightarrow B = 0$

$\to \; u(R,T) = 0$

$A\, J_0(\lambda r) = 0$

$u(r,t) = A\, J_0(\lambda r)\, e^{-\nu \lambda^2 t}$

$u(r,0) = -U_{max}(1 - r^{*2}) = \sum_{k=1}^{\infty} A_k\, J_0(\lambda_k r)$

by multile using $(r^*\, J_0(\lambda_m r^*))$ and integral $\int^1$

$$\int_0^1 -\frac{U}{m\mu}(1-r^{*2})\, r^*\, J_0(\lambda_m r^*)\, dr = \int_0^1 r^* J_0(\lambda_m r^*) \sum_{k=1}^{\infty} A_k J_0(\lambda_n r^*)\, dr$$

but bessel function

$$\int_0^1 x\, J_n(\lambda x)\, J_n(\lambda x)\, dx = \frac{J_{n+1}^2(\lambda x)}{2}$$

$$\frac{J_{n+1}^2(\lambda x)}{2} A_k = -\int_0^1 \frac{U}{m}(1-r^{*2})(r^*)\; J_0(\lambda_m r^*)\, dr$$

$$A_k = \frac{-2}{J_1^2(\lambda x)} \int_0^1 r^*\, J_0(\lambda r^*)\, \frac{U}{m}(1-r^{*2})\, dr$$

$$A_k = \frac{-4\, U_m\, J_2\, \lambda_k}{\lambda_k^2\, J_0^2\, \lambda_k} = \frac{-8\, U_{max}}{\lambda_k^3\, J_0\,(\lambda_k)}$$

$$\text{but} \quad J_2\, \lambda_k = \frac{2}{\lambda_k}\, J_1\,(\lambda_k)$$

$$\Rightarrow \acute{U}_{(r,t)} = A\, J_0\,(\lambda r)\, e^{-\lambda^2 t}$$

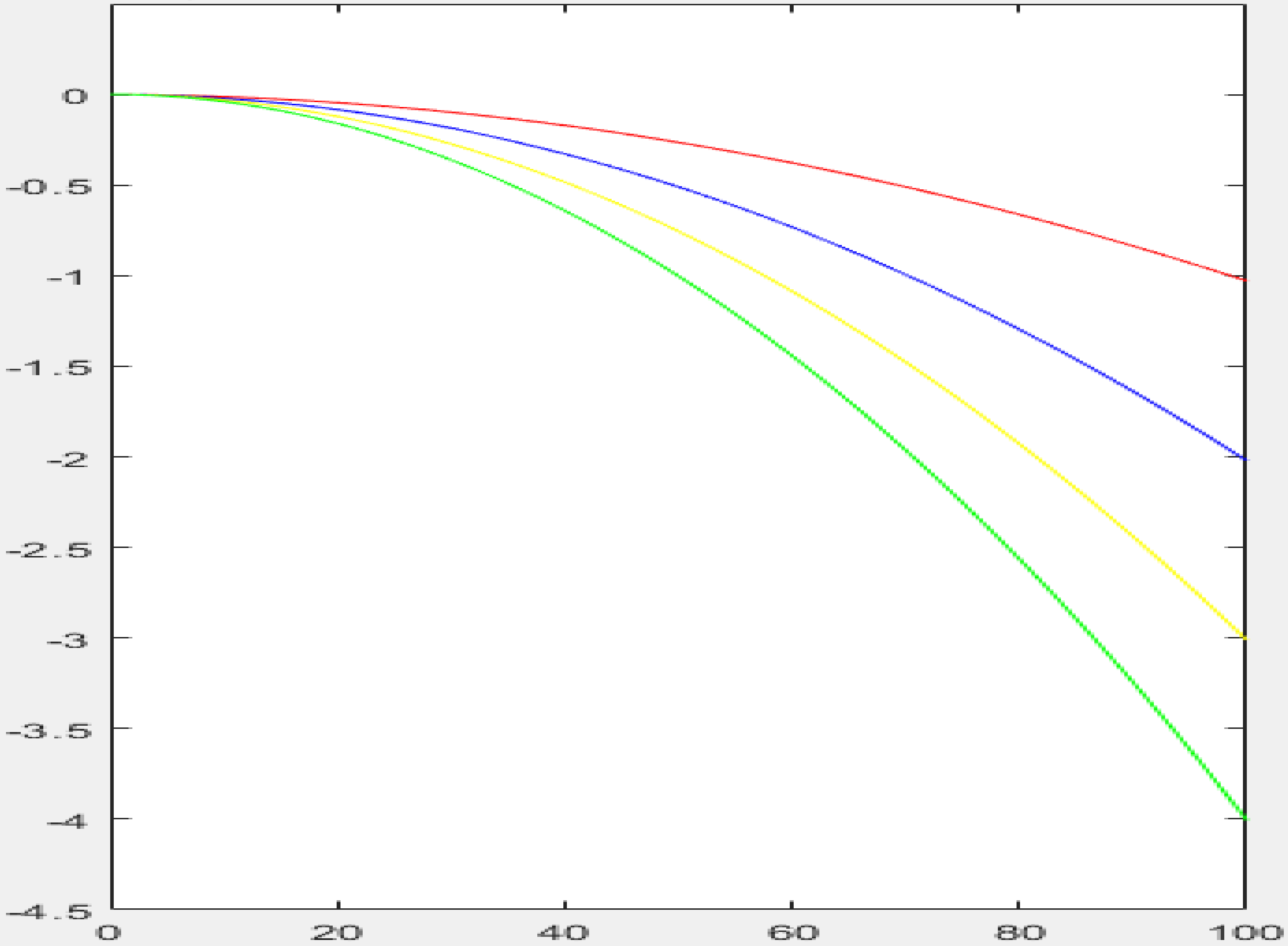
$$u(r,t) = U_m(1 - r^{*2}) - \sum_{k=1}^{\infty} \frac{8\, U_m\, J_0\,(\lambda_k\, r^*)}{\lambda_k^3\, J_1\,(\lambda_k)}\, e^{-\lambda^2 t}$$

Matlab Code

```
clear;
clc;
r=1:0.1:100;
t=0:0.1:100;
Um=1;
J0=1;
J1=1;
k=40;
lamda=2;
lamda=lamda*ones(1,k);
m=min(length(r),length(t));
for ii=1:m
u(ii)=compute_u_r_t(r(ii),t(ii),Um,J0,J1,lamda);
end
figure;
plot3(r(1:m),t(1:m),u,'linewidth',2);
grid on
```

```
function u=compute_u_r_t(r,t,Um,J0,J1,lamda)
u=Um*(1-r.^2);
tmp=8*Um*J0*lamda*r;
tmp=tmp./(lamda.^3);
tmp=tmp/J1;
tmp=tmp./lamda;
tmp=tmp.*exp((-lamda.^2).*t);
sumTmp=sum(tmp);
u=u-sumTmp;
end
```

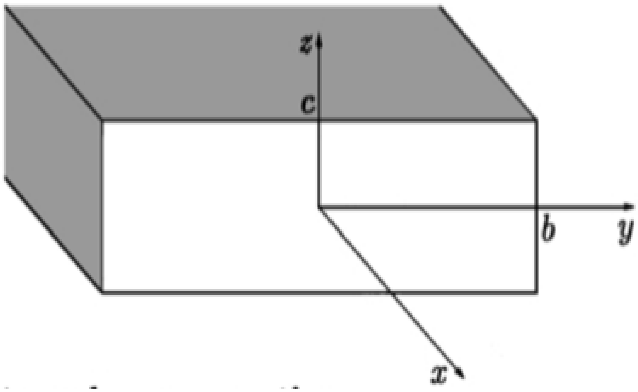
```
function u=compute_u_r_t(r,t,Um,J0,J1,lamda)
u=Um*(1-r.^2);
tmp=8*Um*J0*lamda*r;
tmp=tmp./(lamda.^3);
tmp=tmp/J1;
tmp=tmp./lamda;
tmp=tmp.*exp((-lamda.^2).*t);
sumTmp=sum(tmp);
u=u-sumTmp;
end
```



0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
0
20
40
60
80
100

Poiseuille flow in a tube of rectangular cross section

Poisson equation



$$\frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} = \frac{1}{\mu} \frac{\partial P}{\partial x} \quad \text{-----} \quad (1)$$

boundary condition

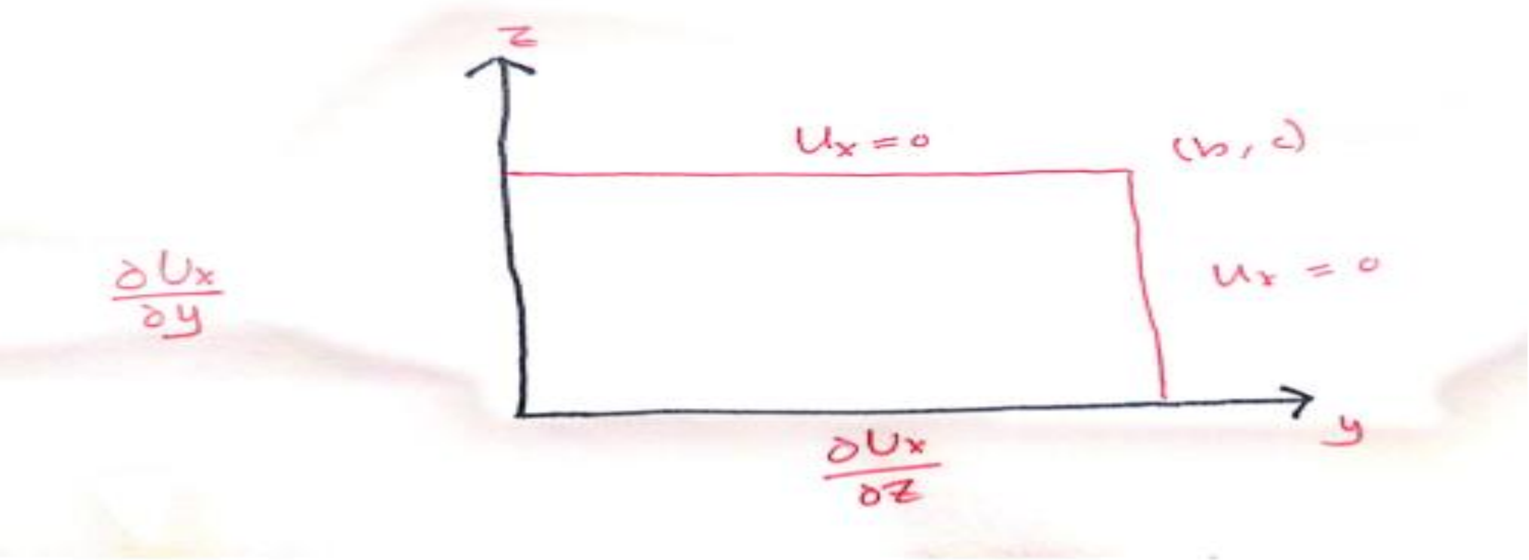
$\rightarrow y = 0 \qquad \frac{\partial U_x}{\partial y} = 0$

$\rightarrow y = b \qquad U_x = 0$

$\rightarrow z = 0 \qquad \frac{\partial U_x}{\partial z} = 0$

$\cancel{\rightarrow U_x = 0}$

$\rightarrow z = c \qquad U_x = 0$



z
$U_x = 0$
(b, c)
$\frac{\partial U_x}{\partial y}$
$U_x = 0$
y
$\frac{\partial U_x}{\partial z}$

$$u(y,z) = \frac{-1}{2\mu} \frac{\partial P}{\partial x} (c^2 - z^2) + u'_x (y,z) \quad \text{---} \ (3)$$

but using equation (3)

→ we can transform equation (1) into laplace equation

also when equation (3) in equation (1) & (2)
weget this equation

$$\frac{\partial^2 u'_x}{\partial y^2} + \frac{\partial^2 u'_x}{\partial z^2} = 0 \quad \text{----} (4)$$

so that boundary condition will be

so that boundary condition will be

$$\frac{\partial u'_x}{\partial y} = 0 \quad @ \quad y = 0$$

$$u'(x) = \frac{1}{2\mu}\frac{\partial p}{\partial x}(c^2 - z^2) \quad @ \quad y = b$$

$$\frac{\partial u'}{\partial z} = 0 \quad @ \quad z = 0$$

$$u'_x = 0 \quad @ \quad z = 0$$

We need to know How we can obtain this solution in detail

so that equation (4) & its boundary condition will solved using seperation of variabels. so that

$$u_x(y,z) = \frac{-1}{2\mu}\frac{\partial p}{\partial x}c^2\left[1-\left(\frac{z}{c}\right)^2+4\sum_{k=1}^{\infty}\frac{(-1)^k}{a_k^3}\frac{\cosh\left(\frac{a_k y}{c}\right)}{\cosh\left(\frac{a_k b}{c}\right)}*\right.$$

$$u_x(y,z) = \frac{-1}{2\mu}\frac{\partial P}{\partial x} c^2 \left[1 - \left(\frac{z}{c}\right)^2 + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{a_k^3} \frac{\cosh\left(\frac{a_k y}{c}\right)}{\cosh\left(\frac{a_k b}{c}\right)} * \cos\left(\frac{a_k z}{c}\right)\right]$$

$$a_k = (2k-1)\frac{\pi}{2} \quad , \quad k = 1, 2, 3, 4, \ldots$$

For the volumetric flow rate

$$Q = \frac{-4}{3\mu}\frac{\partial P}{\partial x} b c^3 \left[1 - \frac{6c}{b}\sum_{k=1}^{\infty} \frac{\tanh\left(\frac{a_k b}{c}\right)}{a_k^5}\right]$$

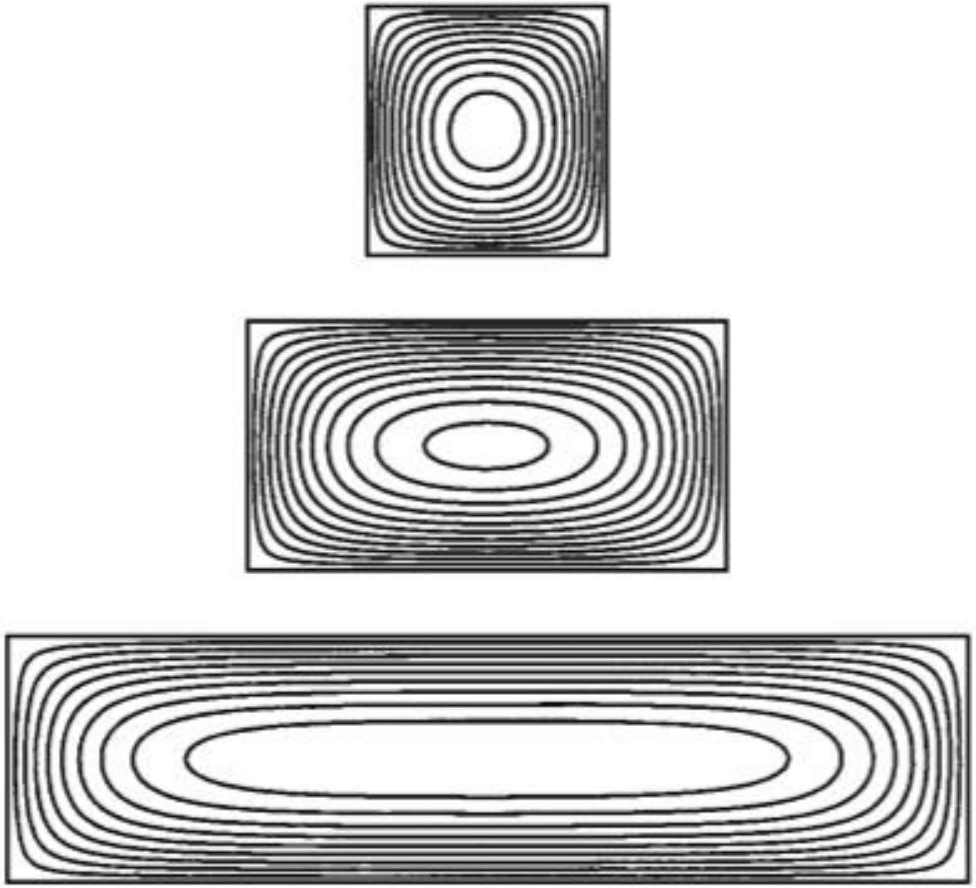


Figure 6.31. *Velocity contours for steady unidirectional flow in tubes of rectangular cross section with width-to-height ratio equal to 1, 2 and 4.*